

P425/1
PURE MATHEMATICS
PAPER 1
JULY 2017
3 HOURS

ST. JOSEPH OF NAZARETH HIGH SCHOOL
UGANDA ADVANCED CERTIFICATE OF EDUCATION
INTERNAL MOCK EXAMINATION 2017
PURE MATHEMATICS
PAPER 1
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INSTRUCTIONS:

- Answer all the questions in Section A and only five questions in Section B.
- Show all necessary working clearly.
- Silent, non-programmable scientific calculators and mathematical tables with a list of formula may be used.

SECTION A (40 MARKS)**Attempt all questions from this section.**

1. If $\frac{2+\sqrt{2}}{2-\sqrt{2}} + \frac{1-\sqrt{2}}{1+\sqrt{2}} = a + b\sqrt{2}$ Find the values of a and b . (5 marks)
2. The ninth term of an arithmetic progression is twice the third term, and the fifteenth term is 27. Evaluate the sum of the first 25 terms of the series. (5 marks)
3. Differentiate $x^{\cos x}$ with respect to x . (5 marks)
4. Evaluate the definite integral $\int_0^1 x \tan^{-1} x \, dx$ (5 marks)
5. Solve the equation $3 \cos 2\theta - 7 \cos \theta - 2 = 0$ for $0^\circ \leq \theta \leq 360^\circ$. (5 marks)
6. Find the equation of the circle which touches the line $3x - 4y = 3$ at the point $(5, 3)$ and passes through the point $(-2, 4)$. (5 marks)
7. The roots of the equation $x^2 + px + 7 = 0$ are α and β . Given that $\alpha^2 + \beta^2 = 22$, find the possible values of p . (5 marks)
8. Prove that $\log_a x = \frac{1}{\log_x a}$. Hence solve the equation $\log_{10} x + \log_x 100 = 3$ (5 marks)

SECTION A (60 MARKS)**Answer any five questions from this section.**

9. (a) If $z = x + iy$, determine the Cartesian equation of the locus given by
$$\left| \frac{(z-1)}{(z+1-i)} \right| = \frac{2}{5}$$
(6 marks)
(b) Sketch the loci defined by the equations:
(i) $\arg(z + 2) = \frac{-2\pi}{3}$
(ii) $\arg\left(\frac{z-3}{z-1}\right) = \frac{\pi}{4}$
(6 marks)
10. (a) Prove that $\sin 4\theta = \frac{4 \tan \theta (1 - \tan^2 \theta)}{(1 + \tan^2 \theta)}$ (6 marks)
(b) Solve the equation $\tan^{-1}(1+x) + \tan^{-1} 1 - x = \frac{\pi}{4}$ (6 marks)

11. Find the coordinates of any maxima, minima and points of inflexion of the function $y = \frac{3x-1}{(4x-1)(x+5)}$ that it may have. Hence sketch the curve $y = \frac{3x-1}{(4x-1)(x+5)}$ (12 marks)

12.(a) Find $\int x\sqrt{(1-x^2)} dx$

(b) Express $\int_0^1 \frac{x^2+x+1}{(x+1)(x^2+1)} dx = \frac{3}{4}\ln 2 + \frac{\pi}{8}$ (9 marks)

13. (a) Find the particular solution of the differential equation $xy \frac{dy}{dx} = x^2 + y^2$,
Given that $y = 2$, when $x = 1$ (6 marks)

(b) A lump of radioactive substance is disintegrating. At time t days after it was first observed to have the mass of 10 grams and $\frac{dm}{dt} = -km$ where k is a constant. Find the time, in days for the substance to reduce to 1 gram in mass, given that its half-life is 10 days. (The half-life is the time in which half of any mass of the substance will decay.) (6 marks)

14. (a) Find the values of m for which the line $y = mx$ is a tangent to the circle $x^2 + y^2 + fy + c = 0$ (3 marks)

- (b) Find the points where the line $2y - x + 5 = 0$ meets the circle $x^2 + y^2 - 4x + 3y - 5 = 0$. Obtain the equation of the tangents and normal to the circle at these points (6 marks)

15. (a) Show that the points A, B and C with position vectors $2\hat{i} + 3\hat{j}$, $4\hat{i} + 5\hat{j}$, $6\hat{i} + 9\hat{j}$ respectively are the vertices of a triangle. Find the area of the triangle. (5 marks)

- (b) Find a vector r perpendicular to the vectors $s = 5\hat{i} + 3\hat{j} + k$ and $t = -\hat{i} + 3\hat{j} + 2k$.

Hence, find the equation of a plane passing through the point $A(5, -1, -2)$ and parallel to s and t . Find the angle between the plane and the line

$$\frac{x-2}{1} = \frac{y-2}{2} = \frac{z-2}{3} \quad (7 \text{ marks})$$

16. (a) If $y = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$ show that $\frac{dy}{dx} = \frac{1}{1+x^2}$ (6 marks)

- (b) Use the Maclaurin's theorem to find the first four terms of the expansion of $e^x \sin x$. (6 marks)

~END~

MARKING GUIDE FOR P42511 INTERNAL MOCK 2017

10.	SOLUTION ST. JONAH'S	MKS	COMMENT	(1)
1.	$\frac{2+\sqrt{2}}{2-\sqrt{2}} + \frac{1-\sqrt{2}}{1+\sqrt{2}} = a+b\sqrt{2}$ <u>Soln</u> $\frac{(2+\sqrt{2})(2+\sqrt{2}) + (1-\sqrt{2})(1-\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2}) (1+\sqrt{2})(1-\sqrt{2})} = a+b\sqrt{2}$ $\frac{4+4\sqrt{2}+2}{4-2} + \frac{1-2\sqrt{2}+2}{1-2} = a+b\sqrt{2}$ $\frac{6+4\sqrt{2}}{2} + \frac{3-2\sqrt{2}}{-1} = a+b\sqrt{2}$ $\Rightarrow 3+2\sqrt{2} - 3+2\sqrt{2} = a+b\sqrt{2}$ $4\sqrt{2} = a+b\sqrt{2}$ $\Rightarrow 0+4\sqrt{2} = a+b\sqrt{2}$ $\therefore a=0 \text{ and } b=4$	M₁M₁ M₁ A₁ A₁ 05	Rationalising / L.C.M	
2.	9th term = $a+8d$ 3rd term = $a+2d$ 15th term = $a+14d$ $\Rightarrow a+8d = 2(a+2d)$ $\cancel{a+14d} = 2$ $a-4d = 0 \quad \dots \textcircled{1}$ $\Rightarrow a+14d = 27 \quad \dots \textcircled{2}$ From $\textcircled{1}$ $a=4d$ Substituting for a in $\textcircled{2}$ $4d+14d = 27$ $18d = 27$ $d = \frac{27}{18} = \frac{3}{2}$ $a = 4 \times \frac{3}{2} = 6$ $S_n = \frac{n}{2} [2a + (n-1)d]$ $S_{25} = \frac{25}{2} [2 \times 6 + 24 \times \frac{3}{2}]$ $= \frac{25}{2} [12 + 36]$ $= \frac{25}{2} \times 48$ $S_{25} = \underline{600}$	B₁ M₁ A₁ M₁ A₁	subst. $a=4d$ in $\textcircled{2}$ for both d and a subst	

Qo.	SOLUTION	MKS	COMMENT
3.	Let $y = x^{\cos x}$ Taking logarithms to base e $\ln y = \ln x^{\cos x}$ $\Rightarrow \ln y = \cos x \ln x$ $\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{d}{dx} (\cos x \ln x)$ $\frac{1}{y} \frac{dy}{dx} = -\sin x \ln x + \frac{\cos x}{x}$ $\frac{1}{y} \frac{dy}{dx} = \frac{\cos x - x \sin x \ln x}{x}$ $\frac{dy}{dx} = y \frac{(\cos x - x \sin x \ln x)}{x}$ $\frac{dy}{dx} = x^{\cos x} \left(\frac{\cos x - x \sin x \ln x}{x} \right)$ $\frac{dy}{dx} = (\cos x - x \sin x \ln x) x^{\cos x - 1}$	M_1 M_1 M_1 B_1 A_1 <u>05</u>	
4.	$\int_0^1 x \tan^{-1} x \, dx$ Let $u = \tan^{-1} x$ $\Rightarrow \tan u = x$ $\sec^2 u \frac{du}{dx} = 1$ $\frac{du}{dx} = \frac{1}{\sec^2 u}$ $\frac{du}{dx} = \frac{1}{1+x^2}$ Let $\frac{dv}{dx} = x$ $v = \int x \, dx$ $v = \frac{x^2}{2}$ ✓	B_1 for $\frac{du}{dx} \times v$	

SOLUTION	MKS	COMMENT
using integration by parts $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$ $\Rightarrow \int x \tan^{-1} x dx = \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$ to $\int \frac{x^2}{1+x^2} dx$ Let $x^2 = \tan^2 \theta$ $x = \tan \theta$ $dx = \sec^2 \theta d\theta$ $\Rightarrow \int \frac{\tan^2 \theta}{1+\tan^2 \theta} \cdot \sec^2 \theta d\theta$ $= \int \tan^2 \theta d\theta$ $= \int (\sec^2 \theta - 1) d\theta$ $\Rightarrow \int \frac{x^2}{1+x^2} dx = \tan \theta - \theta + C$ $\Rightarrow \int \frac{x^2}{1+x^2} dx = x - \tan^{-1} x + C$ substituting in $\int$ $\Rightarrow \int x \tan^{-1} x dx = \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} [x - \tan^{-1} x] + C$ $= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + C$ $\int_0^1 x \tan^{-1} x dx = \left[\frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x \right]_0^1$ $= \left(\frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} \right) - (0)$ $\Rightarrow \int_0^1 x \tan^{-1} x dx = \frac{\pi}{4} - \frac{1}{2} = 0.285$	M₁ B₁ M₁ A₁	substn for $\int \frac{x^2}{1+x^2} dx$
ALF to $\int \frac{x^2}{1+x^2} dx$ By long division $\frac{x^2 + 1 \sqrt{x^2}}{x^2 + 1} = 1 + \frac{-1}{x^2 + 1}$ $\Rightarrow \int \frac{x^2}{1+x^2} dx = \int 1 dx + \int \frac{-1}{x^2 + 1} dx$ $= x - \tan^{-1} x + C$		

(2)

No.	SOLUTION	MKS	COMMENT
5.	$3 \cos 2\theta - 7 \cos \theta - 2 = 0$ $\Rightarrow 3(2 \cos^2 \theta - 1) - 7 \cos \theta - 2 = 0$ $6 \cos^2 \theta - 7 \cos \theta - 2 = 0$ $\text{Let } x = \cos \theta$ $\Rightarrow 6x^2 - 7x - 2 = 0$ $\Rightarrow 6x^2 - 10x + 3x - 2 = 0$ $\Rightarrow 2x(3x - 5) + 1(3x - 5) = 0$ $(2x + 1)(3x - 5) = 0$ $2x + 1 = 0 \text{ \& } 3x - 5 = 0$ $x = -\frac{1}{2} \quad x = \frac{5}{3}$ $\text{For } \cos \theta = -\frac{1}{2}$ $\theta = \cos^{-1}(-\frac{1}{2})$ $\theta = 120^\circ, 240^\circ$ $\text{For } \cos \theta = \frac{5}{3}$ $\theta \text{ is undefined.}$	B₁ M₁ A₁ M₁ B₁	
6.	Using the equation of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ $\text{At point } (5, 3)$ $\Rightarrow 10g + 6f + c = -34 \text{ --- (1)}$ $\text{At point } (-2, 4)$ $-4g + 8f + c = -20 \text{ --- (2)}$ Subs (1) - (2) $14g - 2f = -14$ $\Rightarrow 7g - f = -7 \text{ --- (3)}$	B₁ B₁ for ① & ②	

SOLUTION	MKS	COMMENT
For line $3x - 4y = 3$ its gradient $= \frac{3}{4}$ For $x^2 + y^2 + 2gx + 2fy + c = 0$ $2x + 2y \frac{dy}{dx} + 2g + 2f \frac{dy}{dx} = 0$ $(y + f) \frac{dy}{dx} = -x - g$ $\frac{dy}{dx} = \frac{-x - g}{y + f}$ ✓ At point $(5, 3)$ $\Rightarrow \frac{3}{4} = \frac{-5 - g}{3 + f}$ $\Rightarrow 9 + 3f = -20 - 4g$ $\Rightarrow 4g + 3f = -29$ --- (4) Also $3(3) + (4)$ $21g - 3f = -21$ $+ 4g + 3f = -29$ $25g = -50$ $g = -2$ ✓ Subst $g = -2$ in (3) $-14 - f = -7$ ✓ $f = -7$ Substituting for g and f in (1) $10(-2) + 6(-7) + c = -34$ $\Rightarrow c = 28$ $\therefore$ The eqn of the circle is <u>$x^2 + y^2 - 4x - 14y + 28 = 0$</u> ✓	B₁ M₁ A₁ B₁	(3) for $g, f \& c$

No.	SOLUTION	MKS	COMMENT
7.	$x^2 + px + 7 = 0$ $a + b = -p \text{ \& } ab = 7$ $a^2 + b^2 = (a+b)^2 - 2ab$ $\Rightarrow 22 = p^2 - 14$ $p^2 = 36$ $p = \pm 6$	B1 M1 M1 A1A1	subst (values of p)
8.	$\text{Let } y = \log_a x$ $\Rightarrow a^y = x$ $\log_a a^y = \log_a x$ $y \log_a a = 1$ $y = \frac{1}{\log_a x}$ $\therefore \log_a x = \frac{1}{\log_a x}$ $\log_{10} x + \log_{10} 100 = 3$ $\Rightarrow \log_{10} x + \log_{10} 10^2 = 3$ $\log_{10} x + \frac{2}{\log_{10} x} = 3$ $\text{Let } p = \log_{10} x$ $\Rightarrow p + \frac{2}{p} = 3$ $p^2 + 2 = 3p$ $p^2 - 3p + 2 = 0$ $p^2 - 2p - p + 2 = 0$ $p(p-2) - 1(p-2) = 0$ $(p-1)(p-2) = 0$ $\Rightarrow p = 1 \text{ \& } p = 2$ $\Rightarrow \log_{10} x = 1$ $\Rightarrow x = \underline{10} \checkmark$ $\log_{10} x = 2$ $x = \underline{100} \checkmark$	M1 B1 M1 A1 A1	Taking logs to base x

Q.	SOLUTION	MKS	COMMENT
Q2	$\tan^{-1}(x+1) + \tan^{-1}(1-x) = \frac{\pi}{4}$		
	Let $A = \tan^{-1}(1+x) \Rightarrow \tan A = (1+x)$	B1	
	$B = \tan^{-1}(1-x) \Rightarrow \tan B = (1-x)$	B1	
	$\Rightarrow A + B = \frac{\pi}{4}$		
	$\tan(A+B) = \tan \frac{\pi}{4}$		
	$\frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$	M1	
	$\frac{1+x + 1-x}{1 - (1+x)(1-x)} = 1$	M4	
	$\Rightarrow \frac{2}{1 - (1-x^2)} = 1$		
	$\frac{2}{x^2} = 1$	M1	
	$x^2 = 2$		
	$x = \pm \sqrt{2}$		
	$x = \pm \underline{\underline{1.414}}$	A7	

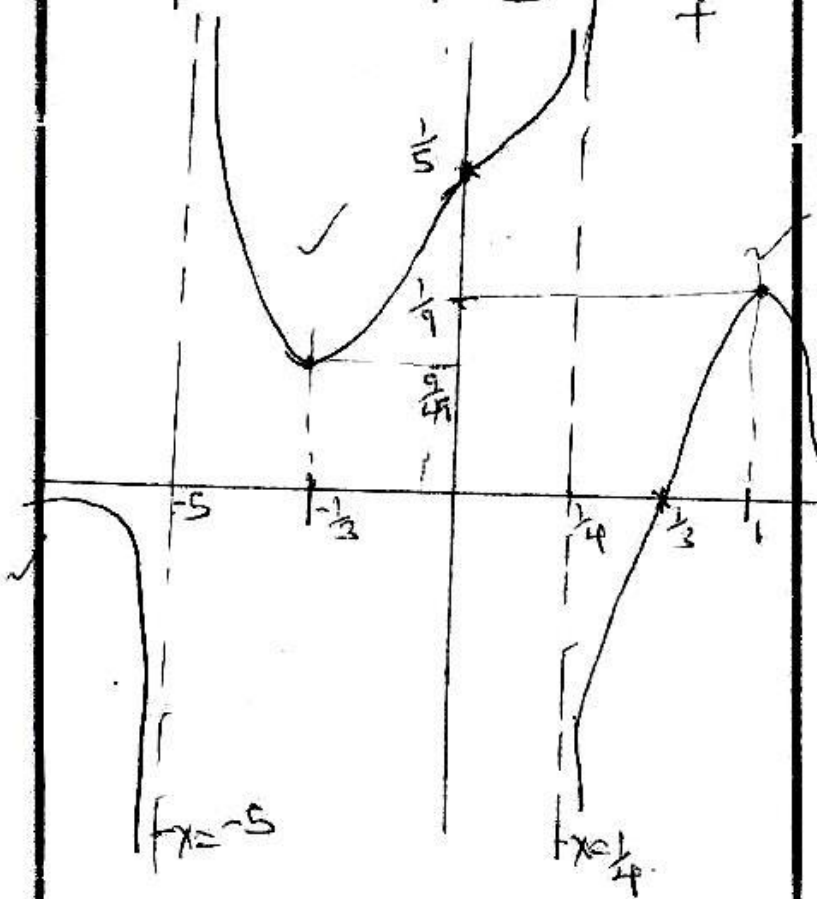
Q.	SOLUTION No. 11	MKS	COMMENT
11	$y = \frac{3x-1}{(4x-1)(x+5)} = \frac{3x-1}{4x^2+19x-5}$ Let $u = 3x-1$, $v = 4x^2+19x-5$ $\frac{du}{dx} = 3 \quad \frac{dv}{dx} = 8x+19$ $\frac{dy}{dx} = \frac{3(4x^2+19x-5) - (3x-1)(8x+19)}{(4x^2+19x-5)^2}$ $= \frac{12x^2+57x-15 - (24x^2+49x-19)}{(4x^2+19x-5)^2}$ $= \frac{-12x^2+8x+4}{(4x^2+19x-5)^2} \quad \checkmark$ At turning points, $\frac{dy}{dx} = 0$ $\Rightarrow \frac{-12x^2+8x+4}{(4x^2+19x-5)^2} = 0$ $\Rightarrow -12x^2+8x+4 = 0$ $3x^2-2x-1 = 0$ $3x^2-3x+x-1 = 0$ $3x(x-1) + 1(x-1) = 0$ $(x-1)(3x+1) = 0$ $x = 1 \text{ or } x = -\frac{1}{3} \quad \checkmark$ when $x = 1$, $y = \frac{3(1)-1}{(4(1)-1)(1+5)} = \frac{2}{9} = 0.22$ when $x = -\frac{1}{3}$, $y = \frac{3(-\frac{1}{3})-1}{(4(-\frac{1}{3})-1)(-\frac{1}{3}+5)}$ $= \frac{-2}{-\frac{4}{3}-1)(\frac{14}{3})} = \frac{9}{49} = 0.18$	M₁ M₁	$\frac{dy}{dx}$

Q.	SOLUTION	MKS	COMMENT
	value of x L $-\frac{1}{3}$ R L 1 R Sign of $\frac{dy}{dx}$ - 0 + + 0 - $\Rightarrow (-\frac{1}{3}, \frac{9}{49})$ is a minimum point ✓ $\Rightarrow (1, \frac{1}{9})$ is a maximum point ✓ for $y = \frac{3x-1}{(4x-1)(x+5)}$ when $x=0$, $y = \frac{-1}{-1 \times 5} = \frac{1}{5}$ The curve cuts the y-axis at $(0, \frac{1}{5})$ ✓ when $y=0$, $\frac{3x-1}{(4x-1)(x+5)} = 0$ $\Rightarrow 3x-1=0$ $\Rightarrow x = \frac{1}{3}$ The curve cuts the x-axis at $(\frac{1}{3}, 0)$ ✓ for vertical asymptotes $(4x-1)(x+5)=0$ $\Rightarrow x = \frac{1}{4}$ and $x = -5$ are the vertical asymptotes. for horizontal asymptotes $y = \frac{3x-1}{4x^2-19x-5} = \frac{\frac{3}{x} - \frac{1}{x^2}}{4 - \frac{19}{x} - \frac{5}{x^2}}$ As $x \rightarrow \pm \infty$, $y \rightarrow \frac{0}{4} = 0$ $y=0$ is the equation for horizontal asymptote.	A1 A1 A1 A1 B1 B1	For vertical asymptotes

SOLUTION	MKS	COMMENT
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- Check for sign of y b/w critical values.

	$x < -5$	$-5 < x < \frac{1}{4}$	$\frac{1}{4} < x < \frac{1}{3}$	$x > \frac{1}{3}$
$3x-1$	-	-	-	+
$4x-1$	-	-	+	+
$x+5$	-	+	+	+
y	-	+	-	+



B₁

B₃ For the three regions with asymptotes properly indicated

12

Q.	SOLUTION	MKS	COMMENT
12.	$\int x \sqrt{1-x^2} dx$ Let $u = 1-x^2$ $\frac{du}{dx} = -2x$ $\frac{du}{dx} \cdot dx = -\frac{du}{2x}$ ✓ $\Rightarrow \int x \sqrt{1-x^2} dx = \int x \sqrt{u} \frac{du}{-2x}$ $= -\frac{1}{2} \int u^{\frac{1}{2}} du$ $= -\frac{2}{3} \left(\frac{1}{2} \right) u^{\frac{3}{2}} + C$ $= -\frac{1}{3} u^{\frac{3}{2}} + C$ ✓ $= -\frac{1}{3} (1-x^2)^{\frac{3}{2}} + C$ ✓ <u>Alternative</u> $\int x \sqrt{1-x^2} dx$ Let $x^2 = \sin^2 u$ $x = \sin u$ $dx = \cos u du$ ✓ $\Rightarrow \int x \sqrt{1-x^2} dx = \int \sin u \sqrt{1-\sin^2 u} \cos u du$ $= \int \sin u \cdot \cos^2 u du$ Let $t = \cos u$ $\frac{dt}{du} = -\sin u$ $du = -\frac{dt}{\sin u}$ $\Rightarrow \int \sin u \cos^2 u du = \int \sin u t^2 \frac{dt}{-\sin u}$ $= -\int t^2 dt$ $= -\frac{1}{3} t^3 + C$ $= -\frac{1}{3} \cos^3 u + C$ But $\sin u = x$ $\Rightarrow \cos u = \sqrt{1-\sin^2 u}$ $\Rightarrow \cos u = \sqrt{1-x^2}$ $\int x \sqrt{1-x^2} dx = -\frac{1}{3} (\sqrt{1-x^2})^3 + C$ ✓ $= -\frac{1}{3} (1-x^2)^{\frac{3}{2}} + C$	M_1 M_1 B_1 M_1 M_1 B_1	

SOLUTION	MKS	COMMENT
Q $\frac{x^2+x+1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$ $\Rightarrow x^2+x+1 = A(x^2+1) + (Bx+C)(x+1)$ put $x = -1 \Rightarrow 1 = 2A$ $\Rightarrow A = \frac{1}{2}$ put $x = 0 \Rightarrow 1 = A + C$ $1 = \frac{1}{2} + C$ $C = \frac{1}{2}$ put $x = 1 \Rightarrow 3 = 2A + (B+C)x$ $3 = 2(\frac{1}{2}) + 2(B+\frac{1}{2})$ $2 = 2B + 1$ $2B = 1$ $B = \frac{1}{2}$ $\therefore \frac{x^2+x+1}{(x+1)(x^2+1)} = \frac{1}{2(x+1)} + \frac{x+1}{2(x^2+1)}$ $\int_0^1 \frac{x^2+x+1}{(x+1)(x^2+1)} dx = \int_0^1 \left(\frac{1}{2(x+1)} + \frac{x+1}{2(x^2+1)} \right) dx$ $= \frac{1}{2} \left[\ln(x+1) \right]_0^1 + \frac{1}{2} \int_0^1 \frac{x+1}{x^2+1} dx$ $= \frac{1}{2} \ln 2 + \frac{1}{2} \int_0^1 \left(\frac{x}{x^2+1} + \frac{1}{x^2+1} \right) dx$ $= \frac{1}{2} \ln 2 + \frac{1}{4} \left[\ln(x^2+1) \right]_0^1 + \frac{1}{2} \left[\tan^{-1} x \right]_0^1$ $= \frac{1}{2} \ln 2 + \frac{1}{4} \ln 2 + \frac{1}{2} \tan^{-1} 1$ $= \frac{1}{4} (2 \ln 2 + \ln 2) + \frac{1}{2} \times \frac{\pi}{4}$ $= \frac{1}{4} (\ln 4 + \ln 2) + \frac{\pi}{8}$ $= \frac{1}{4} \ln 8 + \frac{\pi}{8}$ $= \frac{1}{4} \ln 2^3 + \frac{\pi}{8}$ $= \frac{3}{4} \ln 2 + \frac{\pi}{8}$	M₁ A₁ A₁ A₁ A₁ M₁ M₁ B₁ A₁	

Q.	SOLUTION	MKS	COMMENT
3.	$xy \frac{dy}{dx} = x^2 + y^2$ $\text{Let } y = vx$ $\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$ $\Rightarrow x \cdot vx (v + x \frac{dv}{dx}) = x^2 + v^2 x^2$ $\Rightarrow x^2 v (v + x \frac{dv}{dx}) = x^2 (1 + v^2)$ $\Rightarrow v (v + x \frac{dv}{dx}) = 1 + v^2$ $\Rightarrow v + x \frac{dv}{dx} = \frac{1 + v^2}{v}$ $\Rightarrow x \frac{dv}{dx} = \frac{1 + v^2}{v} - v$ $x \frac{dv}{dx} = \frac{1 + v^2 - v^2}{v}$ $x \frac{dv}{dx} = \frac{1}{v}$ $\Rightarrow v dv = \frac{dx}{x}$ $\int v dv = \int \frac{dx}{x}$ $\frac{1}{2} v^2 = \ln x + C$ $\text{But } y = vx \Rightarrow v = \frac{y}{x}$ $\frac{1}{2} \frac{y^2}{x^2} = \ln x + C$ $\text{when } y = 2, x = 1$ $\frac{1}{2} \left(\frac{4}{1} \right) = \ln 1 + C$ $\Rightarrow C = 2$ $\Rightarrow \frac{y^2}{2x^2} = \ln x + 2$ $\Rightarrow y^2 = 2x^2 (\ln x + 2)$	M₁ B₁ M₁ B₁ A₁ B₁	for reduced/simplified i.e. Substituting for v in general soln.

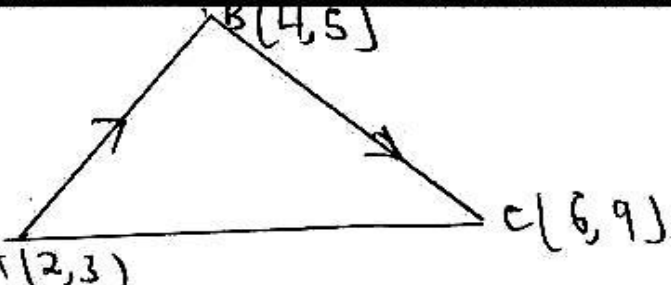
SOLUTION	MKS	COMMENT
$\frac{dm}{dt} = -Km$ $\Rightarrow \frac{dm}{m} = -K dt$ $\int \frac{dm}{m} = \int -K dt$ $\Rightarrow \ln m = -Kt + C \quad \checkmark$	M ₁	
when $t=0, m=10, \Rightarrow C = \ln 10$ $\therefore \ln m = -Kt + \ln 10 \quad \text{--- (1)}$ Half life is 10 days	M ₁	Substituting for C
$\Rightarrow$ when $t=10, m = \frac{1}{2} \times 10 = 5$ grams when $t=10, m=5$g eqn 1 becomes $\ln 5 = -10K + \ln 10$ $10K = \ln 10 - \ln 5$ $10K = \ln 2$ $K = \frac{1}{10} \ln 2 \quad \checkmark$	M ₁	
Eqn 2 becomes $\ln m = -\frac{t}{10} \ln 2 + \ln 10 \quad \text{--- (2)}$ when $m=1, t=?$	-M ₁	Substituting for K
eqn (2) becomes $\ln 1 = -\frac{t}{10} \ln 2 + \ln 10$ $\frac{t}{10} \ln 2 = \ln 10 - \ln 1$ $\frac{t}{10} \ln 2 = \ln 10$ $\frac{t}{10} = \frac{\ln 10}{\ln 2}$	M ₁	Substituting $m=1$
$t = 10 \times \frac{2.3026}{0.693}$ $t = \underline{\underline{33.2 \text{ days}}}$	M ₁	

No.	SOLUTION	MKS	COMMENT
14.	Q. $x^2 + y^2 + fx + c = 0$ --- (1) $y = mx$ --- (2) Substitute for y $\Rightarrow m^2 x^2 + x^2 + f(mx) + c = 0$ $(m^2 + 1)x^2 + mfx + c = 0$ ✓ For a line to be a tangent $b^2 = 4ac$ $4(m^2 + 1)c = (mf)^2$ $4m^2 c + 4c = m^2 f^2$ — $4m^2 c - m^2 f^2 = -4c$ $m^2 (4c - f^2) = -4c$ $m^2 = \frac{-4c}{4c - f^2}$ $m = \pm \sqrt{\frac{-4c}{4c - f^2}}$ $m = \pm \sqrt{\frac{4c}{f^2 - 4c}}$ Q. $2y - x + 5 = 0$ --- (1) $x^2 + y^2 - 4x + 3y - 5 = 0$ --- (2) From (1) $x = 2y + 5$ $\Rightarrow (2y + 5)^2 + y^2 - 4(2y + 5) + 3y - 5 = 0$ ✓ $4y^2 + 20y + 25 + y^2 - 8y - 20 + 3y - 5 = 0$ $5y^2 + 15y = 0$ $5y(y + 3) = 0$ $y = 0$ or $y = -3$ ✓ ∴ points of contact are (5, 0) and (1, -3) ✓ Differentiating $x^2 + y^2 - 4x + 3y - 5 = 0$ $\Rightarrow 2x + 2y \frac{dy}{dx} - 4 + 3 \frac{dy}{dx} = 0$ —	M₁ B₁ A₁ M₁ A₁ A₁ M₁	

Q.

10

SOLUTION	MKS	COMMENT
$(2y+3)\frac{dy}{dx} = 4-2x$ $\frac{dy}{dx} = \frac{4-2x}{2y+3} \checkmark$ At the point $(5,0)$, $\frac{dy}{dx} = \frac{4-2(5)}{2(0)+3} = -2$ Eqn of the tangent $y-0 = (x-5) \cdot -2$ $y = -2x+10 \checkmark$ Eqn of the normal $y-0 = (x-5) \cdot \frac{1}{2}$ $2y = x-5 \checkmark$ At the point $(-1,-3)$, $\frac{dy}{dx} = \frac{4+2}{2(-1)+3} = -2$ Eqn of the tangent $y+3 = (x+1) \cdot -2$ $y+3 = -2x-2$ $y+2x+5 = 0 \checkmark$ Eqn of the normal $y+3 = (x+1) \cdot \frac{1}{2}$ $2y+6 = x+1$ $2y-x+5 = 0 \checkmark$	8) A) A) A) A)	

No.	SOLUTION	MKS	COMMENT
5	2  $A(2,3)$ $B(4,5)$ $C(6,9)$ $\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ $\vec{BC} = \vec{OC} - \vec{OB} = \begin{pmatrix} 6 \\ 9 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ Since $\vec{AB} \neq \lambda \vec{BC}$, therefore the points A, B and C are vertices of a triangle. Area of triangle = $\frac{1}{2} \vec{AB} \times \vec{BC}$ $\vec{AB} \times \vec{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 0 \\ 2 & 4 & 0 \end{vmatrix}$ $= 0\hat{i} - 0\hat{j} + 4\hat{k}$ Area, A = $\frac{1}{2} \times \sqrt{4^2}$ $= \frac{1}{2} \times 4 = 2 \text{ sq units}$ $\vec{r} = \vec{S} \times \vec{t}$ $\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 3 & 1 \\ -1 & 3 & 2 \end{vmatrix}$ $\vec{r} = (3 \times 2 - 3 \times 1)\hat{i} - (5 \times 2 - 1 \times 1)\hat{j} + (5 \times 3 - 1 \times 3)\hat{k}$ $\vec{r} = 3\hat{i} - 11\hat{j} + 18\hat{k}$	B₁ B₁ B₁ M₁ B₁ M₁ A₁	

Q.	SOLUTION	MKS	COMMENT
	$\vec{a} \cdot \vec{p} = p \cdot \vec{q}$ $\begin{pmatrix} 5 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -11 \\ 18 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -11 \\ 18 \end{pmatrix}$ $15 + 11 - 36 = 3x - 11y + 18z$ $-10 = 3x - 11y + 18z$ $\Rightarrow 3x - 11y + 18z + 10 = 0$ Normal vector $\vec{n} = 3\hat{i} - 11\hat{j} + 18\hat{k}$ Let $\vec{d}$ be directional vector for the line $\Rightarrow \vec{d} = \hat{i} + 2\hat{j} + 3\hat{k}$ $\vec{n} \cdot \vec{d} = \vec{n} \vec{d} \cos \theta$ $\begin{pmatrix} 3 \\ -11 \\ 18 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \sqrt{3^2 + (-11)^2 + 18^2} \cdot \sqrt{1^2 + 2^2 + 3^2} \cos \theta$ $3 - 22 + 54 = \sqrt{454} \cdot \sqrt{14} \cos \theta$ $35 = \sqrt{6356} \cos \theta$ $\cos \theta = \frac{35}{\sqrt{6356}}$ $\theta = \cos^{-1} \frac{35}{\sqrt{6356}}$ $\theta = \cos^{-1}(0.439012)$ $\theta = \underline{\underline{26.04^\circ}}$	M_1 A_1 M_1 M_1 A_1	

No.	SOLUTION	MKS	COMMENT
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Q. $y = \sin^{-1} \left(\frac{x}{\sqrt{1+x^2}} \right)$

$\Rightarrow \sin y = \frac{x}{\sqrt{1+x^2}}$

$\cos y \frac{dy}{dx} = \frac{d}{dx} \left(\frac{x}{\sqrt{1+x^2}} \right)$

$\cos y \frac{dy}{dx} = \frac{\sqrt{1+x^2} \frac{d}{dx}(x) - x \frac{d}{dx}(\sqrt{1+x^2})}{1+x^2}$

$\cos y \frac{dy}{dx} = \frac{\sqrt{1+x^2} - \frac{1}{2}x(2x)}{1+x^2}$

$\cos y \frac{dy}{dx} = \left(\frac{1+x^2-x^2}{\sqrt{1+x^2}} \right) \cdot \frac{1}{1+x^2}$

$\cos y \frac{dy}{dx} = \frac{1}{(1+x^2)^{3/2}}$

But $\cos y = \sqrt{1-\sin^2 y}$
 $= \sqrt{1-\frac{x^2}{1+x^2}}$
 $= \sqrt{\frac{1+x^2-x^2}{1+x^2}}$
 $= \frac{1}{\sqrt{1+x^2}}$

Substituting in *
 $\frac{1}{\sqrt{1+x^2}} \cdot \frac{dy}{dx} = \frac{1}{(1+x^2)^{3/2}}$
 $\frac{dy}{dx} = \frac{1}{(1+x^2)^{3/2}} \cdot \sqrt{1+x^2}$
 $\frac{dy}{dx} = \frac{1}{1+x^2}$

M₁

M₁

B₁

B₁

M₁

B₁

for $\frac{d}{dx} \left(\frac{x}{\sqrt{1+x^2}} \right) = \frac{1}{(1+x^2)^{3/2}}$

for $\cos y = \frac{1}{\sqrt{1+x^2}}$

Substituting

Q. SOLUTION	MKS	COMMENT
Q. $f(x) = e^x \sin x$ $\Rightarrow f(0) = e^0 \sin 0 = \underline{0}$ $f'(x) = e^x \sin x + e^x \cos x$ (by prod rule) $\Rightarrow f'(0) = e^0 \sin 0 + e^0 \cos 0 = \underline{1}$ ✓ $f''(x) = e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x$ $= 2e^x \cos x$ $\Rightarrow f''(0) = 2e^0 \cos 0 = \underline{2}$ ✓ $f'''(x) = 2e^x \cos x - 2e^x \sin x$ $\Rightarrow f'''(0) = \underline{2}$ ✓ $f^{(4)}(x) = 2e^x \cos x - 2e^x \sin x - 2e^x \sin x$ $= -4e^x \sin x$ $\Rightarrow f^{(4)}(0) = \underline{0}$ $f^{(5)}(x) = -4e^x \sin x - 4e^x \cos x$ $\Rightarrow f^{(5)}(0) = \underline{-4}$ $\therefore f(x) = x + \frac{2x^2}{2!} + \frac{2x^3}{3!} - \frac{4x^5}{5!}$ $\therefore e^x \sin x = x + x^2 + \frac{x^3}{3} - \frac{1}{30}x^5$	B₁ B₁ B₁ B₁ B₁ B₁	